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MODULES OF VECTOR FIELDS, DIFFERENTIAL FORMS AND DEGENERATIONS OF DIFFERENTIAL SYSTEMS

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ABSTRACT

We deal with $(n-1)$-generated modules of smooth (analytic, holomorphic) vector fields $V=(X_1,\dots,X_{n-1})$ (codimension 1 differential systems) defined locally on $\mathbf{R}^n$ or $\mathbb{C}^n$, and extend the standard duality $(X_1,\dots,X_{n-1})\mapsto(\omega)$, $\omega=\Omega\,(X_1,\dots,X_{n-1},\cdot,)$ (Ω — a volume form) between V's and 1-generated modules of differential 1-forms (Pfaffian equations) — when the generators X_i are linearly independent — onto substantially wider classes of codimension 1 differential systems. We prove that two codimension 1 differential systems V and $(\tilde X_1,\dots,\tilde X_{n-1})$ are equivalent if and only if so are the corresponding Pfaffian equations (ω) and $(\tilde\omega)$ provided that ω has **1-division property**: $\omega\wedge\mu=0$, μ — any 1-form $\Rightarrow\mu=f\omega$ for certain function germ f. The 1-division property of ω turns out to be equivalent to the following properties of V: (a) $fX\in V$, f — not a 0-divisor function germ $\Rightarrow X\in V$ (the **division property**); (b) $(V^\perp)^\perp=V$; (c) $V^\perp=(\omega)$; (d) $(\omega)^\perp=V$, where $\perp$ denotes the passing from a module (of vector fields or differential 1-forms) to its annihilator.

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1. Introduction. Main results

We are interested in the local classification and geometry of singularities of codimension 1 differential systems. These are $(n-1)$-generated modules $V = (X_1, \dots, X_{n-1})$ of smooth (analytic, holomorphic) vector fields in the n-dimensional space $\mathbb{R}^n$ or $\mathbb{C}^n$, over the ring of smooth (analytic, holomorphic) functions. When the generators are pointwise linearly independent, we deal with codimension 1 distribution which can be equivalently described by a single non-vanishing Pfaffian equation $W = (\omega)$, i.e., module of differential 1-forms generated by one 1-form ω. The classification problems can then be parallelly formulated and investigated in either setting. Things change with admitting points at which the generators $X_1, \dots, X_{n-1}$ are linearly dependent. The passing to the language of Pfaffian equations then becomes questionable.

Throughout the paper we use the letter D to denote the class of differentiability. Namely, it will always denote either ∞, or A, or H, that is to say – infinitely smooth (C^∞), real analytic, and holomorphic, respectively.

We denote by $\Lambda^k_{\mathrm{D}}(n)$ (respectively, $V_{\mathrm{D}}(n)$) the set of germs at zero of differential k-forms (respectively, vector fields) of class D on a neighbourhood of zero in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then $\Lambda^k_{\mathrm{D}}(n)$ and $V_{\mathrm{D}}(n)$ are modules over the ring $\Lambda^0_{\mathrm{D}}(n)$ of germs of functions of class D.

Definition: By a local codimension 1 differential system of class D we understand a $\Lambda^0_{\mathrm{D}}(n)$-module $(X_1, \dots, X_{n-1})$ generated by $n-1$ vector fields $X_1, \dots, X_{n-1} \in V_{\mathrm{D}}(n)$ which are independent on an open dense set of a neighbourhood of zero. We will use notation $V^{n-1}_{\mathrm{D}}(n)$ for the set of all local codimension 1 differential systems.

Definition: A local Pfaffian equation of class D is a $\Lambda^0_{\mathrm{D}}(n)$-module (ω) generated by one 1-form $\omega \in \Lambda^1_{\mathrm{D}}(n)$. We will denote the set of all local Pfaffian equations by $P_{\mathrm{D}}(n)$.

Now we define a natural mapping from $V^{n-1}_{\mathrm{D}}(n)$ to $P_{\mathrm{D}}(n)$.

Definition: Given a local codimension 1 differential system $V = (X_1, \dots, X_{n-1})$ of class D, we denote by $\mathbf{i} \circ V$ a local Pfaffian equation of class D generated by a 1-form $\omega \in \Lambda^1_{\mathrm{D}}(n)$ defined by the relation $\omega(\cdot) = \Omega\,(X_1, \dots, X_{n-1}, \cdot)$, where Ω is an arbitrary local volume form of class D and the dot stands for any vector field of $V_{\mathrm{D}}(n)$.

PROPOSITION 1.1: $\mathbf{i}$ *is a well-defined mapping from* $V_{\mathrm{D}}^{n-1}(n)$ *to* $P_{\mathrm{D}}(n)$.

Proof: We have to prove that the choice of Ω and the choice of generators of V are irrelevant. The first is clear, and the second is a direct corollary of the following

LEMMA 1.2 (see [JP, Appendix]): * *Let M be an abstract module over $\Lambda_{\mathrm{D}}^0(n)$ generated by elements $a_1, \dots, a_k$ as well as by $\tilde{a}_1, \dots, \tilde{a}_k$. Then there exists a $k \times k$ matrix H with entries in $\Lambda_{\mathrm{D}}^0(n)$, $\det H(0) \neq 0$, such that the tuple $[\tilde{a}_1, \dots, \tilde{a}_k]$ is obtained from $[a_1, \dots, a_k]$ by right multiplication by H.*

Proof: There exist matrix-valued functions A and B of class D such that $[a_1, \dots, a_k] = [\tilde{a}_1, \dots, \tilde{a}_k]A$ and $[\tilde{a}_1, \dots, \tilde{a}_k] = [a_1, \dots, a_k]B$. It follows that

$$[\tilde{a}_1, \dots, \tilde{a}_k] = [a_1, \dots, a_k](B + (I - BA)C)$$

for any $k \times k$ matrix C (I is the unit matrix). For any two $k \times k$ matrices α and β with real (complex) entries, and in particular for $A(0)$ and $B(0)$, there exists a matrix C with real (complex) entries such that the matrix $\beta + (I - \beta\alpha)C$ is invertible (a proof of this simple fact can be found, for example, in [Mat]), and we can take $H = B + (I - BA)C$. ∎

One of the central questions concerning the passing from codimension 1 differential systems to Pfaffian equations is as follows: under what conditions on $V \in V_{\mathrm{D}}^{n-1}(n)$

$$V \text{ is D-equivalent to } \tilde{V} \in V_{\mathrm{D}}^{n-1}(n) \text{ if and only if} \tag{1.1}$$

$$\mathbf{i} \circ V \text{ is D-equivalent to } \mathbf{i} \circ \tilde{V}.$$

The equivalence of two local differential systems or two local Pfaffian equations is defined according to the general definition of the equivalence of any two sets of germs (of vector fields, 1-forms, functions) Q and $\tilde{Q}$. The D-equivalence of Q and $\tilde{Q}$ means the existence of a local diffeomorphism Φ of class D such that Φ brings any germ of Q to a germ of $\tilde{Q}$, and Φ^{-1} brings any germ of $\tilde{Q}$ to a germ of Q. For the case where Q and $\tilde{Q}$ are $\Lambda_{\mathrm{D}}^0(n)$-modules there is another (equivalent) definition in terms of generators:

* J. Mather [Mat], Proposition on p. 136, is the first to be credited for this lemma, although not in the present setting.

PROPOSITION 1.3 (see [Mat], [JP]): *Two $\Lambda^0_{\mathrm{D}}(n)$-modules $A = (a_1, \ldots, a_k)$ and $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_k)$ (of germs of vector fields, differential 1-forms, functions) are D-equivalent if and only if there exist a local diffeomorphism Φ of class D and a $k \times k$ matrix H with entries in $\Lambda^0_{\mathrm{D}}(n)$, $\det H(0) \neq 0$, such that Φ brings the tuple $[a_1, \ldots, a_k]$ to $[\tilde{a}_1, \ldots, \tilde{a}_k]H$.*

This Proposition is a corollary of Lemma 1.2.

Our first result (Theorem 1.4) gives a condition on a generator of $\mathrm{i} \circ V$ under which (1.1) holds true.

Definition: A germ at zero of a differential 1-form ω has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ if for any 1-form $\tilde{\omega} \in \Lambda^1_{\mathrm{D}}(n)$ such that $\omega \wedge \tilde{\omega} = 0$ there exists a function $f \in \Lambda^0_{\mathrm{D}}(n)$ such that $\tilde{\omega} = f\omega$.

Note that the 1-division property of ω is a property of the Pfaffian equation (ω): by Lemma 1.2 all generators of a Pfaffian equation of class D have or have not 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ simultaneously. We will say that a Pfaffian equation (ω) of class D has 1-division property if ω has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$.

THEOREM 1.4: *Let $V \in V^{n-1}_{\mathrm{D}}(n)$ and $\mathrm{i} \circ V = (\omega)$. Assume that ω has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$. Then (1.1) holds.*

It turns out that 1-division property of $\mathrm{i} \circ V$ is equivalent to some other important properties formulated in Theorem 1.5. In the formulation of these properties we use the operation $\perp$: given a set Q of germs of vector fields (differential 1-forms) of class D, we denote by $Q^\perp$ the set of germs of differential 1-forms (vector fields) of class D annihilating all vector fields (annihilated by all 1-forms) from Q.

THEOREM 1.5 (MAIN THEOREM): *Let $V \in V^{n-1}_{\mathrm{D}}(n)$, $\mathrm{i} \circ V = (\omega)$. The following properties of V are equivalent:*

(a) $(\omega)^\perp = V$,

(b) $(\omega) = V^\perp$,

(c) $(V^\perp)^\perp = V$,

(d) *(the* **division property** *of V): if $f \in \Lambda^0_{\mathrm{D}}(n)$ is not a 0-divisor, $X \in V_{\mathrm{D}}(n)$ and $fX \in V$ then $X \in V$,*

(e) *ω has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$.*

Theorem 1.4 is a simple corollary of Main Theorem. In fact, take $V \in V_{\mathrm{D}}^{n-1}(n)$, and let $\mathrm{i}\circ V = (\omega)$. Assume that V satisfies the condition of Theorem 1.4, i.e., (e) holds. Take any $\tilde{V} \in V_{\mathrm{D}}^{n-1}(n)$ such that $\mathrm{i}\circ\tilde{V} = (\tilde{\omega})$. By Main Theorem V and $\tilde{V}$ satisfy (a), that is, $(\omega)^{\perp} = V$, $(\tilde{\omega})^{\perp} = \tilde{V}$. These relations imply Theorem 1.4 since the D-equivalence of any two sets of germs of vector fields or differential 1-forms implies the D-equivalence of their annihilators (via the inverse of the local diffeomorphism giving the former equivalence).

The 1-division property of $\mathrm{i}\circ V$ implies one more property of V — the maximality.

THEOREM 1.6: *Let $V \in V_{\mathrm{D}}^{n-1}(n)$, $\mathrm{i}\circ V = (\omega)$. Assume that ω has 1-division property in $\Lambda_{\mathrm{D}}^{0}(n)$. Then V is not included in any module $\tilde{V} \in V_{\mathrm{D}}^{n-1}(n)$.*

This Theorem is also a simple corollary of Main Theorem. Assume that $V \subset \tilde{V}$, $\tilde{V} \in V_{\mathrm{D}}^{n-1}(n)$. Then at a generic point x (such that $\dim V(x) = n-1$) $V(x) = \tilde{V}(x)$, and therefore $\tilde{V}(x) \subset \ker\omega|_x$. It follows that $\omega|_x$ annihilates any vector of $\tilde{V}(x)$ for any point x, i.e., $\tilde{V} \subset (\omega)^{\perp}$. By Main Theorem $(\omega)^{\perp} = V$, and we obtain $V = \tilde{V}$.

By Main Theorem each of the properties (a) through (e) implies the maximality of V. The inverse is not true: there are maximal differential systems $V \in V_{\mathrm{D}}^{n-1}(n)$ violating each of (a) - (e).

Example: Consider a differential system

$$V = \left(x_1\frac{\partial}{x_1} + x_3\frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3}\right) \in V_{\mathrm{D}}^{2}(3).$$

The Pfaffian equation $\mathrm{i}\circ V$ is generated by the 1-form $x_3(x_3dx_1 - x_1dx_2 - x_2dx_3)$. This 1-form has no 1-division property, and by Main Theorem each of the conditions (a) - (e) is violated. Nevertheless, one can prove that V is maximal.

In view Theorems 1.4 - 1.6 it is relevant to know how to check whether a 1-form ω has 1-division property. Corresponding results are given (with proofs) in the next section. These results allow us to prove that the germ at any point of a generic $(n-1)$-generated module of vector fields on an n-dimensional manifold satisfies the properties (a) - (e) formulated in Main Theorem (section 3). In section 4 we fully characterize which Pfaffian equations come out as $\mathrm{i}\circ V$, where V is a codimension 1 differential system having the simplest degeneration at 0 — the dropping of dimension by 1. Main Theorem is proved in section 5.

Finally, in section 6 are presented some instances of specific applicability of the developed methods to classification of involutive distributions with singularities and geometric properties of global generic codimension 1 differential systems.

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2. 1-Division property

In this section we use the technique of [Mou, ch.1] where the division property of 1-forms is studied. The division property of ω is stronger than the 1-division property; it means that for any $k = 1, \ldots, n-1$ and any differential k-form η such that $\omega \wedge \eta = 0$ there exists a $(k-1)$-form θ such that $\eta = \omega \wedge \theta$.

At first we associate, to any 1-form $\omega \in \Lambda^1_{\mathrm{D}}(n)$, an ideal $I(\omega)$ in $\Lambda^0_{\mathrm{D}}(n)$ and show that the 1-division property of ω is a property of $I(\omega)$.

Let $\omega \in \Lambda^1_{\mathrm{D}}(n)$. Take any local coordinate system $x = (x_1, \ldots x_n)$ of class D. In this coordinate system $\omega = f_1(x)dx_1 + \cdots + f_n(x)dx_n$. Denote by $I(\omega)$ the ideal in $\Lambda^0_{\mathrm{D}}(n)$ generated by $f_1(x), \ldots, f_n(x)$.

PROPOSITION 2.1:

(i) *The ideal $I(\omega)$ is invariantly related to ω, i.e., does not depend on a coordinate system.*

(ii) *Assume that the ideals $I(\omega)$ and $I(\tilde{\omega})$ are D-equivalent. Then ω and $\tilde{\omega}$ have or have not 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ simultaneously.*

Proof: (i) Let $f_1(x), \ldots, f_n(x)$ be the coefficients of ω in a coordinate system x of class D, $I(\omega) = (f_1(x), \ldots, f_n(x))$. Let y be another coordinate system related to the coordinate system x via a local diffeomorphism Φ of class D: $x = \Phi(y)$. The tuple of the coefficients of $\Phi^*\omega$ is equal to $[f_1(\Phi(y)), \ldots, f_n(\Phi(y))]\Phi'(y)$. Since $\Phi'(y)$ is invertible, $I(\Phi^*\omega)$ is generated by functions $f_1(\Phi(y)), \ldots, f_n(\Phi(y))$ and we see that $I(\Phi^*\omega) = \Phi^* I(\omega)$.

(ii) It is easy to verify that if $\omega = f_1(x)dx_1 + \cdots + f_n(x)dx_n \in \Lambda^1_{\mathrm{D}}(n)$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ then:

(a) for any $n \times n$ matrix $H(x)$ with entries in $\Lambda^0_{\mathrm{D}}(n)$ and such that $\det H(0) \neq 0$, the 1-form with the tuple of coefficients $[f_1(x), \ldots, f_n(x)]H(x)$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$.

Here, for certain Φ, $I(\tilde{\omega}) = \Phi^* I(\omega) = I(\Phi^*\omega)$ (by (i)). On applying Lemma 1.2 to this ideal, and using (a) for $\Phi^*\omega$, (ii) follows. ∎

THEOREM 2.2:

(1) *Let $D = A$ or H. The 1-division property of $\omega \in \Lambda^1_{\mathrm{D}}(n)$ in $\Lambda^0_{\mathrm{D}}(n)$ is equivalent to the following property:*

(f) *the formal series of the coefficients of ω in some (and then any) coordinate system have no common factor with zero free term.*

(2) *Let $\omega \in \Lambda^1_\infty(n)$. Assume that $I(\omega)$ is smoothly equivalent to an ideal in $\Lambda^0_\infty(n)$ generated by analytic germs.* Then the 1-division property of ω in $\Lambda^0_\infty(n)$ is equivalent to* (f).

This theorem implies the following corollary for codimension 1 differential systems.

THEOREM 2.3 (corollary of Theorem 2.2 and Main Theorem): *Let $V \in V^{n-1}_{\mathrm{D}}(n)$ and $\mathrm{i}\circ V = (\omega)$.*

(1) *If $D = A$ or H then the properties* (a) - (e) *formulated in Main Theorem are equivalent to* (f).

(2) *If $D = \infty$ and $I(\omega)$ is smoothly equivalent to an ideal in $\Lambda^0_\infty(n)$ generated by analytic germs then the properties* (a) - (e) *are equivalent to* (f).

The second statements of Theorems 2.2 and 2.3 are not true for arbitrary codimension 1 differential systems:

Example: see [Mou]. Let $A = e^{-1/x_1^2}$, $B = x_2$, $\omega = A\,dx_1 + B\,dx_2 \in \Lambda^1_\infty(n)$. The formal series of the coefficients of ω have non-trivial common factor x_2, but it is easy to check that ω has 1-division property. Note now that $(\omega) = \mathrm{i} \circ V$ where $V \in V^{n-1}_{\mathrm{D}}(n)$, $V = (A\frac{\partial}{\partial x_2} - B\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_n})$. By Main Theorem V satisfies (a) through (e), but ω does not satisfy (f).

OPEN QUESTION: *We do not know whether* (f) *implies 1-division property in $\Lambda^0_\infty(n)$ for arbitrary 1-form $\omega \in \Lambda^1_\infty(n)$.*

Proof of Theorem 2.2: Fix a coordinate system $x = (x_1, \dots, x_n)$. Denote by $\Lambda^0_{\mathrm{FR}}(n)$ (respectively $\Lambda^0_{\mathrm{FC}}(n)$) the ring of formal series in $x_1, \dots, x_n$ with real (respectively complex) coefficients. Let F = FR or FC. By $\Lambda^k_{\mathrm{F}}(n)$ we denote the module of differential k-forms with coefficients in $\Lambda^0_{\mathrm{F}}(n)$. We will say that $\omega \in \Lambda^1_{\mathrm{F}}(n)$ has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$ if for any 1-form $\tilde\omega \in \Lambda^0_{\mathrm{F}}(n)$ such that $\omega \wedge \tilde\omega = 0$ there exists a formal series $q \in \Lambda^0_{\mathrm{F}}(n)$ such that $\tilde\omega = q\,\omega$.

* This property does not hold only for ideals of a codimension ∞ subset in the set of all n-generated ideals; see [Mal].

Given a germ $f = f(x) \in \Lambda^0_{\mathrm{D}}(n)$ we will denote by $\hat{f}$ its formal series; given a 1-form $\omega = f_1(x)dx_1 + \cdots + f_n(x)dx_n$, $f_i \in \Lambda^0_{\mathrm{D}}(n)$ we will denote by $\hat{\omega}$ a 1-form $\hat{f}_1 dx_1 + \cdots + \hat{f}_n dx_n \in \Lambda^1_{\mathrm{F}}(n)$ (F = FR if D = A, ∞; F = FC if D = H). We will say that a 1-form $\omega \in \Lambda^1_{\mathrm{D}}(n)$ has **1-division property** in $\Lambda^0_{\mathrm{F}}(n)$ if $\hat{\omega}$ has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$.

The first statement of Theorem 2.2 is a corollary of the two following Lemmas:

LEMMA 2.4: *Let $D = A$ or H. A 1-form $\omega \in \Lambda^1_{\mathrm{D}}(n)$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ if and only if it has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$ ($F = FR$ if $D = A$, $F = FC$ if $D = H$).*

LEMMA 2.5: *Let $F = FR$ or FC, $\omega = q_1(x)dx_1 + \cdots + q_n(x)dx_n \in \Lambda^1_{\mathrm{F}}(n)$. Then ω has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$ if and only if the formal series $q_1, \ldots, q_n$ have no non-trivial common factor (i.e., common factor with zero free term).*

Proof of Lemma 2.4: (i) At first assume that $\omega \in \Lambda^1_{\mathrm{D}}(n)$ has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$. Take any 1-form $\eta \in \Lambda^1_{\mathrm{D}}(n)$ such that $\omega \wedge \eta = 0$. Then $\hat{\omega} \wedge \hat{\eta} = 0$, and therefore there exists $q \in \Lambda^0_{\mathrm{F}}(n)$ such that $\hat{\eta} = q\hat{\omega}$. We use the following property of analytic and holomorphic function germs: if A and B are analytic (holomorphic) function germs and the formal series of A is divisible by a formal series of B, then A is divisible by B, and A/B is an analytic (holomorphic) function germ. By this property q is the formal series of an analytic (holomorphic) function germ, whence the 1-division property of ω in $\Lambda^0_{\mathrm{D}}(n)$.

(ii) Now we prove that if $\omega \in \Lambda^1_{\mathrm{D}}(n)$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ then it has 1-division property in $\Lambda^0_{\mathrm{F}}(n)$. 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ means that the sequence

$$\Lambda^0_{\mathrm{D}}(n) \xrightarrow{\wedge\omega} \Lambda^1_{\mathrm{D}}(n) \xrightarrow{\wedge\omega} \Lambda^2_{\mathrm{D}}(n) \tag{2.1}$$

is exact. We have to prove the exactness of the sequence

$$\Lambda^0_{\mathrm{F}}(n) \xrightarrow{\wedge\hat{\omega}} \Lambda^1_{\mathrm{F}}(n) \xrightarrow{\wedge\hat{\omega}} \Lambda^2_{\mathrm{F}}(n). \tag{2.2}$$

The exactness of (2.2) follows from that of (2.1) since the ring $\Lambda^0_{\mathrm{F}}(n)$ is flat over the ring $\Lambda^0_{\mathrm{D}}(n)$; see [Mal]. ∎

Proof of Lemma 2.5: (i) If the formal series $q_1, \ldots, q_n$ have non-trivial common factor q then $\omega = q\eta$, $\eta \in \Lambda^1_{\mathrm{F}}(n)$, and $\omega \wedge \eta = 0$. Assuming that $\eta = \gamma\omega$, $\gamma \in \Lambda^0_{\mathrm{F}}(n)$ we obtain that $(q\gamma - 1)\omega = 0$, which is only possible if $\omega = 0$ (since the formal series $q\gamma$ has the free term -1). The contradiction shows that ω does not have 1-division property in $\Lambda^0_{\mathrm{F}}(n)$.

(ii) Assume now that the formal series $q_1, \ldots, q_n$ have no non-trivial common factor. Without lost of generality we can assume that $q_1 \neq 0$. Take any 1-form $\eta = \tilde{q}_1 dx_1 + \cdots + \tilde{q}_n dx_n \in \Lambda^1_{\mathrm{F}}(n)$ such that $\omega \wedge \eta = 0$. Take any formal series θ with zero free term which is an irreducible factor of q_1. There exists $j \in \{2, \ldots, n\}$ such that θ is not a factor of q_j. Since $q_1 \tilde{q}_j = q_j \tilde{q}_1$, the series θ is a factor of $\tilde{q}_1$ (because in the ring of formal series every irreducible factor is prime; see [Mal]). It means that any irreducible factor of q_1 is a factor of $\tilde{q}_1$. The ring of formal series is factorial (see [Mal]), therefore $\tilde{q}_1 = \gamma q_1$ for some formal series γ. The relations $q_1 \tilde{q}_j = q_j \tilde{q}_1$ imply $\tilde{q}_j = \gamma q_j$, $\gamma \in \Lambda^0_{\mathrm{F}}(n)$, $j = 1, \ldots, n$, i.e. $\eta = \gamma \omega$, whence the 1-division property of ω in $\Lambda^0_{\mathrm{F}}(n)$. ∎

The first statement of Theorem 2.2 is proved. The second statement can be reduced, due to Proposition 2.1, to the following

LEMMA 2.6: *Let $\omega \in \Lambda^1_{\mathrm{A}}(n)$. Then ω has 1-division property in $\Lambda^0_{\mathrm{A}}(n)$ if and only if it has 1-division property in $\Lambda^0_{\infty}(n)$.*

The proof is exactly the same as that of Lemma 2.4; we use the flatness of the ring $\Lambda^0_{\infty}(n)$ over the ring $\Lambda^0_{\mathrm{A}}(n)$; see [Mal].

3. Generic codimension 1 differential systems on a manifold

Properties (a) – (e) formulated in Main Theorem hold, of course, for generic germs $V \in V^{n-1}_{\mathrm{D}}(n)$ (a generic germ V is generated by pointwise independent vector fields). One can expect that they also hold for any germ of a generic globally defined codimension 1 differential system. The following theorem says that it is true.

THEOREM 3.1: *In the space of all $(n-1)$-tuples of global vector fields of class D on an n-manifold M there exists an open dense set such that for any tuple $[X_1, \ldots, X_{n-1}]$ in this set and for any $p \in M$ the local differential system V generated by germs of $X_1, \ldots, X_{n-1}$ at p satisfies (a) – (e).*

The main tool to prove this theorem is the following

LEMMA 3.2: *Let $[X_1, \ldots, X_{n-1}]$ be a generic tuple of vector fields of class D on an n-manifold M. Let $p \in M$, and let*

$$k = n - 1 - \dim \operatorname{span}(X_1, \ldots, X_{n-1})(p) \geq 1. \tag{3.1}$$

Let V be a local codimension 1 differential system of class D generated by germs at p of $X_1, \ldots, X_{n-1}$. There exist generators $Y_1, \ldots, Y_{n-1}$ of V and a coordinate system of class D near p such that

$$
\begin{aligned}
Y_i &= \frac{\partial}{\partial x_i} + \sum_{s=n-k}^{n} b_{i,s}(x)\frac{\partial}{\partial x_s}, \quad i = 1, \ldots, n-k-1, \\
Y_j &= \sum_{s=n-k}^{n} a_{j,s}(x)\frac{\partial}{\partial x_s}, \quad j = n-k, \ldots, n-1,
\end{aligned}
\tag{3.2}
$$

where the $k(k+1)$ function germs $a_{j,s}$ are differentially independent.

Proof: It is easy to see that any local codimension 1 differential system satisfying (3.1) admits a tuple of generators of the form (3.2). Here the degeneration "$n-1$ vectors span a linear space of dimension $n-1-k$" has codimension $k(k+1)$ (by the Corank Product lemma; see [AGV]), and we can exclude typically all such k that $k(k+1) > n$ (by Transversality theorem, [AGV]). Now the degeneration "$k(k+1)$ algebraic 1-forms are dependent" has codimension $n+1-k(k+1)$, and the degeneration violating the conclusion of the lemma has codimension $n+1$. To conclude the proof we can again use Transversality theorem. ∎

Proof of Theorem 3.1: Let Ω be the standard volume form $dx_1 \wedge \cdots \wedge dx_n$. In view of Lemma 3.2 and Main Theorem, it suffices to prove that the 1-form $\omega = \Omega(Y_1, \ldots, Y_{n-1}, \cdot)$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$. Let $f_1, \ldots, f_n$ be the coefficients of ω. Note that the 1-division property of ω follows from that of $f_r dx_r + f_s dx_s$, where r, s is any pair of different indices provided that f_r is not a 0-divisor in $\Lambda^0_{\mathrm{D}}(n)$. In fact, if $\omega \wedge \tilde{\omega} = 0$, where $\tilde{\omega} = \tilde{f}_1 dx_1 + \cdots + \tilde{f}_n dx_n$, then $(f_r dx_r + f_s dx_s) \wedge (\tilde{f}_r dx_r + \tilde{f}_s dx_s) = 0$, and the 1-division property of $f_r dx_r + f_s dx_s$ implies that $\tilde{f}_r = h f_r$ for a certain function $h \in \Lambda^0_{\mathrm{D}}(n)$. Now the relations $f_r \tilde{f}_i = f_i \tilde{f}_r$, $i = 1, \ldots, n$ imply $f_r(\tilde{\omega} - h\omega) = 0$, and, since f_r is not a 0-divisor, $\tilde{\omega} = h\omega$ which means the 1-division property of ω.

We will check that the 1-form $\mu = f_{n-k} dx_{n-k} + f_n dx_n$ has 1-division property and its coefficients are not 0-divisors; by the observation above it implies Theorem 3.1. The differential independence of $a_{j,s}$ implies the existence of a D-diffeomorphism Φ such that

$$[a_{n-k,n-k}, \ldots, a_{n-k,n}, \ldots, a_{n-1,n-k}, \ldots, a_{n-1,n}](\Phi) = [x_1, \ldots, x_{k(k+1)}].$$

Then $f_{n-k}(\Phi)$ and $f_n(\Phi)$ are certain polynomials P_1 and P_2. Therefore f_{n-k} and f_n are not 0-divisors; it remains to check the 1-division property of μ. By

Theorem 2.2 it suffices to show that P_1 and P_2 have no common non-trivial factor within formal series. The latter can be easily checked for arbitrary n and k (for example, if $n = 6$, $k = 2$ then $P_1 = x_2x_6 - x_3x_5$, and $P_2 = x_1x_5 - x_2x_4$). ∎

4. Description of Im i

When the dimension of a module V at the origin drops by two or more then the image of $\mathbf{i}$ seems to be fairly complicated. On the other hand, it is possible to describe Pfaffian equations coming in this way from modules having the first occurring singularities — when $\dim V(0) = n - 2$.

THEOREM 4.1:

(i) *For any* $V \in V_{\mathrm{D}}^{n-1}(n)$ *such that* $\dim V(0) = n - 2$ *there exist linearly independent* $\omega_1, \omega_2 \in \Lambda_{\mathrm{D}}^1(n)$ *and function germs* $f_1, f_2 \in \Lambda_{\mathrm{D}}^0(n)$ *such that*

$$\mathbf{i} \circ V = (f_1\omega_1 + f_2\omega_2), \tag{4.1}$$

$$|f_1|^2 + |f_2|^2 \neq 0 \text{ on an open dense subset of a neighbourhood of zero.} \tag{4.2}$$

(ii) *Let* $f_1, f_2 \in \Lambda_{\mathrm{D}}^0(n)$ *be germs satisfying* (4.2), ω_1 *and* ω_2 *be independent 1-forms in* $\Lambda_{\mathrm{D}}^1(n)$. *There exists* $V \in V_{\mathrm{D}}^{n-1}(n)$ *such that* $\dim V(0) = n - 2$ *and* (4.1) *holds.*

Proof: (i) Take a submodule $\tilde{V}$ of V generated by $n - 2$ pointwise independent vector fields. There exist two independent at 0 differential 1-forms $\omega_1, \omega_2 \in \Lambda_{\mathrm{D}}^1(n)$ such that $\tilde{V}^{\perp} = (\omega_1, \omega_2)$. Then $V^{\perp} \subset (\omega_1, \omega_2)$, and, all the more so, $\mathbf{i} \circ V \subset (\omega_1, \omega_2)$. Therefore (4.1) holds with some functions $f_1, f_2 \in \Lambda_{\mathrm{D}}^0(n)$. These functions satisfy (4.2), since the generators of V are independent on an open dense subset of a neighbourhood of zero (by the definition of codimension 1 differential system).

(ii) The set of all vector fields annihilated by ω_1 and ω_2 is an $(n-2)$-generated module $(Y_1, \ldots, Y_{n-2})$, $Y_1, \ldots, Y_{n-2}$ independent at 0 vector fields. Let Z_1 and Z_2 complete $Y_1, \ldots, Y_{n-2}$ to a basis, and be such that $\omega_i(Z_j) = \delta_{ij}$ (the Kronecker delta). Put $X := f_2Z_1 - f_1Z_2$. Take arbitrary local volume form Ω of class D and put

$$\tilde{\Omega} = \frac{\Omega}{\Omega(Y_1, \ldots, Y_{n-2}, Z_1, Z_2)},$$
$$\tilde{\omega} = \tilde{\Omega}(Y_1, \ldots, Y_{n-2}, X, \cdot).$$

Then

$$\tilde{\omega}(Y_i) = (f_1\omega_1 + f_2\,\omega_2)(Y_i) = 0, \quad i = 1, \ldots, n-2,$$
$$\tilde{\omega}(Z_j) = (f_1\omega_1 + f_2\,\omega_2)(Z_j) = f_j, \quad j = 1, 2.$$

Therefore

$$\tilde{\omega} = f_1\omega_1 + f_2\,\omega_2,$$

whence (4.1) holds with $V = (Y_1, \ldots, Y_{n-2}, X)$. By (4.2), V is a local codimension 1 differential system. ∎

In view of Theorem 4.1 it is relevant to give a condition — on a couple $[f, g]$ — for a 1-form $\omega = f\omega_1 + g\,\omega_2$ to have 1-division property.

We will say that a couple $[f_1, f_2] \in (\Lambda^0_{\mathrm{D}}(n))^2$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ if the relation $f_1\tilde{f}_2 = f_2\tilde{f}_1$, $[\tilde{f}_1, \tilde{f}_2] \in (\Lambda^0_{\mathrm{D}}(n))^2$ implies the existence of $\gamma \in \Lambda^0_{\mathrm{D}}(n)$ such that $[\tilde{f}_1, \tilde{f}_2] = \gamma\,[f_1, f_2]$. If $\mathrm{D} = \mathrm{A}, \mathrm{H}$ or $\mathrm{D} = \infty$ and the ideal (f_1, f_2) is smoothly equivalent to an ideal generated by analytic function germs, then by Theorem 2.2 the 1-division property of $[f_1, f_2]$ in $\Lambda^0_{\mathrm{D}}(n)$ is equivalent to the property "the formal series of f_1 and f_2 have no common factor with zero free term".

PROPOSITION 4.2: *Let $\omega_1, \omega_2 \in \Lambda^1_{\mathrm{D}}(n)$ be independent 1-forms, $f_1, f_2 \in \Lambda^0_{\mathrm{D}}(n)$ be functions satisfying* (4.2). *The 1-form $\omega = f_1\omega_1 + f_2\,\omega_2$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ if and only if the couple $[f_1, f_2]$ has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$.*

Proof: (i) Assume that $[f_1, f_2]$ has 1-division property. We have to prove the 1-division property of ω. Let μ be a 1-form such that

$$\omega \wedge \mu = 0. \tag{4.3}$$

Take vector fields $Z_1, Z_2 \in V_{\mathrm{D}}(n)$ such that $\omega_i(Z_j) = \delta_{ij}$. Then, substituting Z_1, Z_2 into (4.3) we have $f_1\mu(Z_2) = f_2\,\mu(Z_1)$, and it follows from the 1-division property of $[f_1, f_2]$ that $\mu(Z_2) = f_1\,\gamma$ and $\mu(Z_1) = f_2\,\gamma$ for some function germ $\gamma \in \Lambda^0_{\mathrm{D}}(n)$. Using this, by substitution of only Z_2 into (4.3) we obtain $f_2\,(\mu - \gamma\,\omega) = 0$. Similarly, substituting Z_1 into (4.3) we obtain $f_1\,(\mu - \gamma\,\omega) = 0$. It follows from (4.2) that $\mu = \gamma\,\omega$, whence the 1-division property of ω.

(ii) Assume now that ω has 1-division property. To show that $[f_1, f_2]$ has 1-division property take any tuple $[\tilde{f}_1, \tilde{f}_2]$ such that $f_1\tilde{f}_2 = f_2\tilde{f}_1$ and put $\tilde{\omega} := \tilde{f}_1\omega_1 + \tilde{f}_2\,\omega_2$. Then $\omega \wedge \tilde{\omega} = 0$, and $\tilde{\omega} = \gamma\,\omega$ for some function germ γ. Since ω_1 and ω_2 are independent, we obtain $[\tilde{f}_1, \tilde{f}_2] = \gamma\,[f_1, f_2]$, whence the 1-division property of $[f_1, f_2]$. ∎

5. Proof of Main Theorem

The scheme of the proof is as follows:

$$(b) \Rightarrow (e) \Rightarrow (a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (b).$$

The implications (b)$\Rightarrow$(e), (a)$\Rightarrow$(c), (c)$\Rightarrow$(d) and (e)$\Rightarrow$(b) are almost obvious, and we begin with them.

(b)$\Rightarrow$(e). Assume that $\mu \in \Lambda^1_{\mathrm{D}}(n)$ and $\omega \wedge \mu = 0$. Take arbitrary $X \in V$. Then $\mu(X)\omega = 0$, and -- since ω does not vanish on an open dense set of a neighbourhood of zero — we obtain that $\mu(X) = 0$. It means that $\mu \in V^{\perp}$. By (b) $\mu \in (\omega)$, i.e., $\mu = f\omega$ for a certain function $f \in \Lambda^0_{\mathrm{D}}(n)$.

(a)$\Rightarrow$(c). Note that $V \subset (V^{\perp})^{\perp}$ under no assumptions. Also, $(\omega) \subset V^{\perp}$ (under no assumptions), whence $(V^{\perp})^{\perp} \subset (\omega)^{\perp} = V$.

(c)$\Rightarrow$(d). Let $fX \in V$, f not a 0-divisor. Then $\mu(fX) = 0$ for any $\mu \in V^{\perp}$. Since f is not a 0-divisor, $\mu(X) = 0$, i.e., $X \in (V^{\perp})^{\perp} = V$.

(e)$\Rightarrow$(b). $(\omega) \subset V^{\perp}$ under no assumptions. We have to prove that $V^{\perp} \subset (\omega)$. Take $\mu \in V^{\perp}$. The 2-form $\omega \wedge \mu$ vanishes at any point x such that $\dim V(x) = n-1$. Consequently, $\omega \wedge \mu = 0$, and by (e) $\mu \in (\omega)$.

Now we prove the implication (e)$\Rightarrow$(a).

We have to prove that if $X \in (\omega)^{\perp}$ then $X \in V$. Take any volume form Ω of class D and define the following 1-forms:

$$\omega_1 = \Omega(X_2, X_3, \dots, X_{n-1}, X, \cdot),$$
$$\omega_2 = \Omega(X_1, X_3, \dots, X_{n-1}, X, \cdot),$$
$$\dots$$
$$\omega_{n-1} = \Omega(X_1, X_2, \dots, X_{n-2}, X, \cdot),$$

where $X_1, \dots, X_{n-1}$ are generators of V. It is easy to check that $(\omega \wedge \omega_i)(X_j, \cdot) = 0$ for any i, $j = 1, \dots, n-1$. The vector fields $X_1, \dots, X_{n-1}$ are independent on an open dense set in a neighbourhood of zero, therefore $\omega \wedge \omega_i = 0$, $i = 1, \dots, n-1$.

By 1-division property there exist functions $f_1, \dots, f_{n-1}$ of class D such that $\omega_i = f_i\omega$. Consider a vector field $\tilde{X} = X + (-1)^{n-1} f_1 X_1 + (-1)^{n-2} f_2 X_2 + \cdots - f_{n-1} X_{n-1}$. Define 2-forms

$$\theta_1 = \Omega(X_2, X_3, \dots, X_{n-1}, \cdot, \cdot),$$
$$\theta_2 = \Omega(X_1, X_3, \dots, X_{n-1}, \cdot, \cdot),$$
$$\dots$$
$$\theta_{n-1} = \Omega(X_1, X_2, \dots, X_{n-2}, \cdot, \cdot).$$

Compute

$$\theta_i(\tilde{X},\cdot) = \theta_i(X,\cdot) + (-1)^{n-i} f_i \theta_i(X_i,\cdot) = \omega_i - f_i\omega = 0. \tag{5.1}$$

If p is a generic point of a neighbourhood of zero then $\theta_i(p)$ is a non-zero algebraic 2-form with the kernel $\mathrm{span}(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_{n-1})(p)$. The intersection of these kernels is $\{0\}$, hence, by (5.1), $\tilde{X}(p) = 0$. Therefore $\tilde{X} = 0$, that is, $X \in V$.

It remains to prove the implication (d)⇒(e). Let $\omega \wedge \mu = 0$, $\mu \in \Lambda^1_{\mathrm{D}}(n)$. Then

$$\omega \wedge \mu(Z,\cdot) = \omega(Z)\mu - \mu(Z)\omega = 0 \quad \forall \text{ vector field } Z. \tag{5.2}$$

Since ω does not vanish at a generic point of a neighbourhood of zero there exists a vector field Z such that $f = \omega(Z) \neq 0$ at a generic point of a neighbourhood of zero. Let $g := \mu(Z)$. Relation (5.2) assumes the form

$$f\mu = g\omega. \tag{5.3}$$

Our aim is to justify

LEMMA 5.1: *g is divisible by f in $\Lambda^0_{\mathrm{D}}(n)$.*

On assuming this claim, the rest of the reasoning (d)⇒(e) is straightforward: there exists a function $\varphi \in \Lambda^0_{\mathrm{D}}(n)$ such that $g = \varphi f$, so that (5.3) takes the form $f(\mu - \varphi\omega) = 0$. Since f does not vanish at a generic point of a neighbourhood of zero we obtain $\mu - \varphi\omega = 0$, so ω has 1-division property.

Proof of Lemma 5.1: The main tool is the following known identity valid for any vector fields $Z_1,\ldots,Z_{n+1}$ on $\mathbb{R}^n$ or $\mathbb{C}^n$ (Ω — any volume form)

$$\begin{aligned}\Omega(Z_1,\ldots,Z_n)Z_{n+1} &- \Omega(Z_1,\ldots,Z_{n-1},Z_{n+1})Z_n \\ &+ \Omega(Z_1,\ldots,Z_{n-2},Z_n,Z_{n+1})Z_{n-1} - \cdots \\ &+ (-1)^n\Omega(Z_2,\ldots,Z_{n+1})Z_1 = 0.\end{aligned} \tag{5.4}$$

At first we apply this identity to the vector fields $gX_1, X_2, \ldots, X_{n-1}, v_1, v_2$, where v_1 and v_2 are arbitrary vector fields. In this way we get $g\omega(v_1)v_2 - g\omega(v_2)v_1 \in (X_1,\ldots,X_{n-1})$, i.e., by (5.3) $f\mu(v_1)v_2 - f\mu(v_2)v_1 \in (X_1,\ldots,X_{n-1})$. Putting $Q := \mu(v_1)v_2 - \mu(v_2)v_1$, we have $fQ \in (X_1,\ldots,X_{n-1})$. The function f is not a 0-divisor, and by hypothesis (d) $Q \in (X_1,\ldots,X_{n-1})$, i.e., $Q = r_1X_1 + \cdots +$

$r_{n-1}X_{n-1}$ for some functions $r_1, \dots, r_{n-1}$ of class D; hence $fQ = f\,r_1X_1 + \cdots + f\,r_{n-1}X_{n-1}$. But fQ has another expression resulting from (5.4), being also a linear combination of the generators of V. In particular, comparing coefficients at X_{n-1} we get $\Omega\,(gX_1, X_2, \dots, X_{n-2}, v_1, v_2) = -\,f\,r_{n-1}$ at at a generic point of a neighbourhood of zero. Hence this equality holds in a full neighbourhood of zero, i.e.

$$\text{(5.5)}\qquad \begin{array}{l}\Omega\,(gX_1, X_2, \dots, X_{n-2}, v_1, v_2) \text{ is divisible by } f \text{ in } \Lambda^0_{\mathrm{D}}(n)\\ \text{for every vector fields } v_1, v_2.\end{array}$$

We take now arbitrary vector field v_3 and apply (5.4) to the vector fields

$$gX_1, X_2, \dots, X_{n-2}, v_1, v_2, v_3,$$

knowing by (5.5) that all the functions $\Omega\,(gX_1, X_2, \dots, X_{n-2}, v_i, v_j)$, $1 \le i < j \le 3$ are divisible by f in $\Lambda^0_{\mathrm{D}}(n)$. Using the same arguments as previously, it follows that all $\Omega\,(gX_1, X_2, \dots, X_{n-3}, v_1, v_2, v_3)$ are divisible by f in $\Lambda^0_{\mathrm{D}}(n)$. Repeating this procedure and increasing in each step the number of v's, we eventually deal with vector fields $gX_1, v_1, \dots, v_n$ and get that $\Omega(v_1, \dots, v_n)\,g$ is divisible by f. On taking independent vector fields $v_1, \dots, v_n$, and having the factor $\Omega(v_1, \dots, v_n)$ invertible, the lemma is proved. ∎

6. Application of the Pfaffian equation language to geometry of vector fields' modules

We are going to demonstrate some instances of those applications. In use will be the geometric locus of singularities of a given $(n-1)$-generated module V of vector fields on a C^∞ manifold M^n of dimension n, $M_{\mathrm{deg}}(V) := \{p \in M^n \mid \dim V(p) < n-1\}$, as well as another important singularity set associated to V via its Pfaffian equation $(\omega) = \mathrm{i} \circ V$, $M_{\mathrm{sing}}(V) := \{p \in M^n \mid \omega \wedge (d\omega)^k|_p = 0\}$, where $k = \left[\frac{n-1}{2}\right]$. Note that $M_{\mathrm{deg}}(V) \subset M_{\mathrm{sing}}(V)$ and outside $M_{\mathrm{sing}}(V)$, V defines a contact (if n is odd) or quasicontact (if n is even) structure, i.e., generic (of course, nonintegrable) field of hyperplanes (see [AGi], [Ma]).

The structure of the set $M_{\mathrm{sing}}(V)$ depends on the dimension n. For example, if $n = 3$ then $M_{\mathrm{sing}}(V)$ is a smooth surface and $M_{\mathrm{deg}}(V)$ is a smooth curve provided that V is generated by generic vector fields; see [JP]. If $n = 4$ then $M_{\mathrm{deg}}(V)$ is a smooth 2-surface, while $M_{\mathrm{sing}}(V)$ is a stratified manifold; the strata are $M_{\mathrm{deg}}(V)$

and a curve intersecting $M_{\mathrm{deg}}(V)$ at isolated points. If $n \geq 5$ then the structure of $M_{\mathrm{sing}}(V)$ is more complicated; see [MZh].

In most cases the passing to the Pfaffian equation allows one to obtain these and other results on singularities of codimension 1 differential systems much more effectively than using the language of vector fields only. As an example we give here a proof of the following result.

THEOREM 6.1 ([JP], 5.2, (3)): *Let V be a generic smooth codimension 1 differential system on a 3-manifold, p a generic point of $M_{\mathrm{deg}}(V)$. Then the germ of V at p is equivalent to*

$$\left(\frac{\partial}{\partial x},\ x\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right). \tag{6.1}$$

Using the passing $V \to \mathbf{i} \circ V = (\omega)$, the genericness conditions of [JP] can be formulated as follows:

(1) $j^1_p(\omega \wedge d\omega) \neq 0$ and, therefore, $M_{\mathrm{sing}}(V)$ is a smooth surface near p and $d\omega(p) \neq 0$;

(2) $\ker d\omega(p)$ is transversal to $M_{\mathrm{sing}}(V)$.

Proof of Theorem 6.1: We first normalize ω. Since $d\omega(p) \neq 0$, we can use Darboux theorem and reduce the germ of ω at p to the form $xdy + dH(x, y, z)$ where $j^1_0 H = 0$, $p = (0,0,0)$. In these coordinates $M_{\mathrm{sing}}(V) = \{\frac{\partial H}{\partial z} = 0\}$, and $\ker d\omega = (\frac{\partial}{\partial z})$. It follows from (1) and (2) that $\frac{\partial^2 H}{\partial z^2}(0) \neq 0$. By a change of the coordinate z (and multiplying, if necessary, by -1) H can be reduced to $z^2 + h(x, y)$ (we use the Morse lemma with parameters, see [AGV]), whence ω is reducible to the form $zdz + \tilde{\omega}$, where $\tilde{\omega} = A(x, y)dx + B(x, y)dy$. Since $\dim V(p) = 1$ (otherwise $j^1_p\omega = 0$), there exists a vector field v, such that $v(p) \neq 0$ and $\omega(v) = 0$. Let

$$v = a(x, y, z)\frac{\partial}{\partial x} + b(x, y, z)\frac{\partial}{\partial y} + c(x, y, z)\frac{\partial}{\partial z}.$$

Then

$$a(x, y, z)A(x, y) + b(x, y, z)B(x, y) + zc(x, y, z) \equiv 0. \tag{6.2}$$

As $A(0) = B(0) = 0$, it follows that $c(0) = 0$, and, since now $M_{\mathrm{sing}}(V) = \{z = 0\}$, we obtain that $v(p) \in T_p M_{\mathrm{sing}}(V)$ (this phenomenon is called in [JP] a strange typical nongenericness). Substituting $z = 0$ in (6.2) we obtain $a(x,y,0)A(x,y) + b(x,y,0)B(x,y) \equiv 0$. Since $v(p) \neq 0$ and $c(0) = 0$, either $a(0) \neq 0$ or $b(0) \neq 0$. Therefore $\tilde{\omega} = f(x,y)\theta$, where f is a function vanishing at p and θ is a 1-form, $\theta(p) \neq 0$. Now we can reduce $\tilde{\omega}$ to the form $g(x,y)dy$, and — since $d\tilde{\omega}(p) \neq 0$ — to xdy. Thus (ω) is equivalent to $(zdz + xdy)$. The form $zdz + xdy$ has 1-division property and, by Theorem 1.4, V is equivalent to $(zdz+xdy)^\perp = (\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$. ∎

The mentioned massing can also be effectively applied to involutive distributions. Under the involutive distribution we understand any k-generated Lie algebra of vector fields, i.e., a module $V = (X_1, \dots, X_k)$ closed with respect to the Lie bracket: $[X_i, X_j] \in V$, $i,j = 1, \dots, k$. Results of the present paper are in particular applicable to the transition from involutive distributions of codimension 1 ($k = n-1$) to integrable 1-forms (forms ω such that $\omega \wedge d\omega = 0$). Within the scope of the division property, the classification of one kind of these objects reduces to the classification of the other kind of objects.

Consider, for example, a codimension 1 involutive distribution V of class D generated by vector fields $X_1, \dots, X_{n-1}$ on $\mathbb{R}^n$ with independent 1-jets of the form

$$j_0^1 X_i = \lambda_{i1} x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_{in} x_n \frac{\partial}{\partial x_n},$$

$i = 1, \dots, n-1$. Then the Pfaffian equation $(\omega) := \mathrm{i} \circ V$ is generated by a 1-form ω such that

$$j_0^{n-1}\omega = a_1 x_2 x_3 \cdots x_n \, dx_1 + a_2 x_1 x_3 \cdots x_n \, dx_2 + \cdots + a_n x_1 x_2 \cdots x_{n-1} \, dx_n$$

(so-called logarithmic 1-form, cf. [CeLN], [CaLN]), where the vector $[a_1, \dots, a_n] \neq 0$ is orthogonal to the vectors $[\lambda_{i1}, \dots, \lambda_{in}]$, $i = 1, \dots, n-1$. Under the condition $a_i \neq 0$, $i = 1, \dots, n$ ω has 1-division property in $\Lambda^0_{\mathrm{D}}(n)$ (one can use Theorem 2.2). Therefore, by Theorem 1.4, within this case, the classification of V's boils down to that of the respective Pfaffian equations.

For instance, under certain extra genericness conditions on $a_1, \dots, a_n$, ω is D-equivalent to $j_0^{n-1}\omega$ (see [CeLN]). This together with the results of the present paper imply that under those conditions on $a_1, \dots, a_n$, V is equivalent to $j_0^1 V$. That is to say, in suitable coordinates V is then generated by linear diagonal vector fields.

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